Stochastic resonance in bistable systems subject to signal and quasimonochromatic noise

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A bistable system subject to a periodic signal and quasimonochromatic noise (narrow band) is investigated. The system is experimentally realized by a set of linear and nonlinear electric devices, and stochastic-resonance-like responses of the system to the input are found. Under the optimal match of the control parameters the signal-to-noise ratio of the output may be higher than that of the input by a factor of 10, if the signal-to-noise ratio is defined in terms of power spectra.

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In the recent decade the topic of stochastic resonance (SR) has attracted much attention in the nonlinear science community [1–10]. A model most extensively investigated is a bistable system subject to sinusoidal signal and white noise

$$\dot{x}(t)=x-x^3+U\cos\omega t+\Gamma(t)\ ,\qquad \langle\Gamma(t)\rangle=0,\quad \langle\Gamma(t)\Gamma(t')\rangle=2D\delta(t-t')\ . \tag{1}$$

The interplay between the periodic force, white noise, and bistable system has been analyzed in a great detail. However, a problem of theoretical significance and practical importance, how various noise parts at different frequencies play a role in this interplay, especially, how the noise part with spectra close to the input frequency (which will be called the same frequency noise, i.e., the SFN) plays a role, is completely unsolved to date. The study of SFN is extremely interesting, since noise with frequency far from the signal frequency can be easily eliminated by a linear filter while it is much more difficult to eliminate SFN by applying linear treatments. Therefore it will become a great advantage if the SR device can reduce SFN.

A major portion of investigations on SR have focused on the study of the signal-to-noise ratio (SNR), because SNR represents the quality of a signal which plays a central role in the information transfer. Considerable efforts have been devoted to answer the problem of whether the SR devices can enhance SNR better than a linear filter, i.e., whether the SNR of the output from a SR device can be higher than that of the input. To date, all answers to this problem by considering model (1) are negative.

In this paper, we propose a model of a bistable system subject to a periodic signal and quasimonochromatic noise. By quasimonochromatic noise we mean zero-mean Gaussian noise with a power spectrum having a narrow Lorentzian peak centered at a certain finite frequency. An electric circuit set is designed to realize this system. A stochastic-resonance response of the system to the input is found. In experiment we get an output with SNR much higher than that of the input under optimal conditions.

To investigate the interplay of quasimonochromatic noise with the periodic signal and the bistable system, we replace Eq. (1) by the following model:

$$\dot{x}(t)=x-x^3+z(t)\ , \tag{2a}$$

$$\ddot{y}(t)+r\dot{y}(t)+(2\pi f)^2y(t)=U\cos(2\pi f_0t)+H(t)\ , \tag{2b}$$

$$z(t)=ky(t)\ . \tag{2c}$$

We put $f=f_0$ afterwards unless specified otherwise. In Eq. (2c) we use a linear amplifier to flexibly adjust the strength of the input in (2a) to reach the optimal condition. It is an easy matter to realize system (2) by using a set of electric circuits represented in Fig. 1, where LF is a two-pole linear filter representing (2b), LA is a linear amplifier in which k can be freely varied in the relevant range, SRD is the bistable system of (2a) of which the detailed structure is given in Ref. [11] (and in many similar SR analog simulation sets). M's are one-pole Butterworth low-frequency-passing filters for avoiding the aliasing effect. In our experiment we read the signal $U\cos(2\pi f_0t)$ and noise H directly from the signal and noise generators, respectively. The effective voltage H is obviously proportional to $\sqrt{D}$. The concrete proportion coefficient is not relevant to our problem, and not specified. In the following we fix $f_0=120$ Hz and vary U, H, k, and the quality factor of LF, $Q=2\pi f_0/r$, to investigate the system response to the input $I(t)=U\cos(2\pi f_0t)+H(t)$.

The system (2) (or the device Fig. 1) is rather practical.

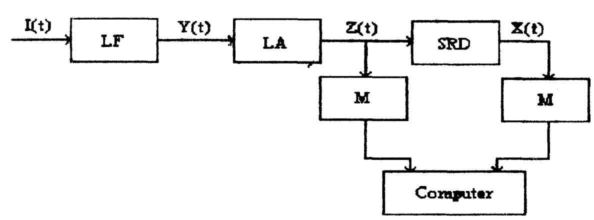

FIG. 1. The block of the experimental set.

It represents reasonable ideas in dealing with signal and noise. Given an input we often first use linear filters to rule out noise. In case linear devices cannot help, we invoke nonlinear treatments to further enhance SNR. It is interesting to ask whether the combinative use of linear filter and bistable system can reach SNR higher than that obtained by separately using a linear filter or a bistable device alone.

Conventionally, the SNR ($\mathcal{R}$) is defined by the ratio of the signal power to the noise power of unit background spectra at the input frequency. For convenience, in experiments one often uses spectra of measured time series [e.g., the measured data of $\overline{I(t)}$] instead of the power spectra to compute SNR ($\overline{\mathcal{R}}$); it is obvious that $\overline{\mathcal{R}} \propto \sqrt{\mathcal{R}}$. Throughout the paper we will consider $\overline{\mathcal{R}}$, and use the notation $\mathcal{R}$ instead of $\overline{\mathcal{R}}$. Experimentally, the SNR is measured as the ratio of the spectrum height at the input frequency to the average height of the side spectra in the vicinity of the input frequency which is regarded as the height of the SFN spectrum. This definition works well as the background noise spectra around f_0 are flat. However, with a linear filter the noise spectrum has a sharp peak at f_0, and the above approach turns out to be inconvenient.

In this paper, we will use an alternative definition of SNR. We make N ($N=100$ for our plots) identical experiments, and get the spectra of the time series in experiments. Then we measure the height of the spectrum at f_0 only, which is denoted by A_i for the ith experiment. We regard the average $\langle A \rangle = \sum_{i=1}^{N} A_i / N$ and the fluctuation

$$\sqrt{\langle \delta A^2 \rangle} = \frac{1}{N} \left[\sum_{i=1}^{N} (A_i - \langle A \rangle)^2 \right]^{1/2}$$

as the signal amplitude and noise strength, respectively. Now SNR is defined by the inverse relative fluctuation

$$\mathcal{R} = \langle A \rangle / \sqrt{\langle \delta A^2 \rangle} \ . \tag{3}$$

For each experiment we use total integration time $T=1$ second and sampling frequency $\nu=2048$ per second. In Figs. 2(a) and 2(b) we fix $U=0.02$ V, $H=0.7$ V, and plot SNR of $z(t)$ ($\mathcal{R}_L$) vs k and Q, respectively. Both curves are horizontal lines. The SNR cannot be changed by either linear amplifier (varying k), or linear filter (varying Q) in the given ranges. This feature is reasonable since the SNR defined in (3) takes account of only the SFN, and represents the intrinsic quality irrelevant to linear treatments.

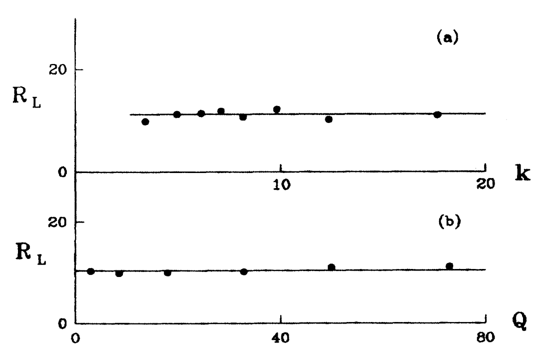

FIG. 2. R_L against k (a) and Q (b). $f_0=120$ Hz (the same in all the following figures), $U=0.02$ V, $H=0.7$ V. Linear filter and amplifier do not change R_L.

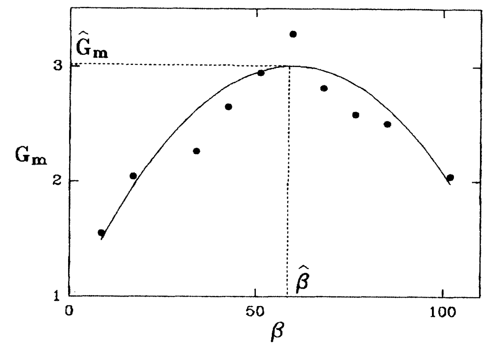

FIG. 3. G_m vs β, $\beta=\sqrt{2}H/U$, $Q=73$. At $\hat{\beta}$ G_m gets its maximum $\hat{G}_m$.

We denote the SNR's of the output $x(t)$ and the input $z(t)$ [or, equally, $y(t)$ or $I(t)$] by $R_{\rm SR}$ and R_L, respectively. The key point is whether we can get $R_{\rm SR} > R_L$, namely, whether the quantity

$$G = R_{\rm SR} / R_L \tag{4}$$

can be larger than 1. The new SNR, defined in (3), is proportional to the standard SNR though the absolute values of the new and old SNR's are different. Then the ratio (4) computed by using the new definition of SNR is identical to that obtained by using the standard SNR definition. This identity is verified in experiments as the noise spectrum around f_0 is flat. As the noise spectrum is sharply peaked at f_0 the large fluctuation and inaccuracy in measurement of the average height of the side spectra makes the standard approach unpractical. Then we have to use the new SNR definition to compute G. Without a linear filter, i.e., for Eq. (1), we find $G \lesssim 1$ whatever U and H. However, the situation is dramatically changed for finite Q in Eqs. (2).

We proceed with our experiment as follows: Given a combination of Q, U, and H, we first change k, measure G for each k, and get G_m, the largest G with respect to k.

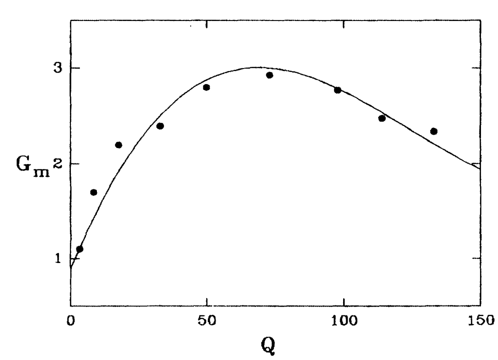

FIG. 4. G_m vs Q. $\beta=50$.

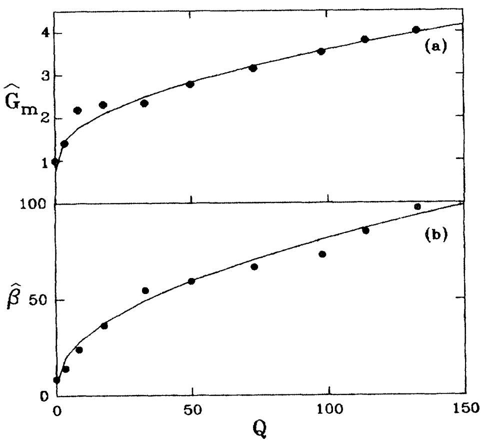

FIG. 5. $\hat{G}_m$ vs Q (a) and $\hat{\beta}$ vs Q (b).

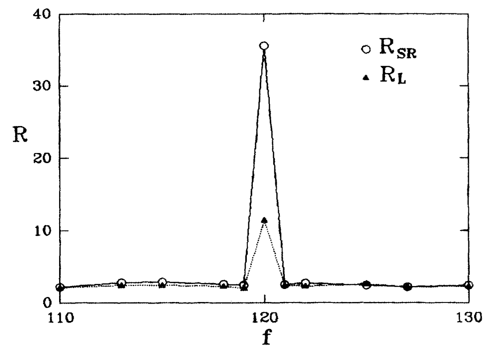

FIG. 6. R_{SR} and R_L vs f. $Q=73$, $\beta=60$. Both curves have peaks at $f=f_0$. R_{SR} is much higher than R_L at $f=f_0$.

Then we vary H to get the corresponding G_m. In Fig. 3 we plot G_m against $\beta=\sqrt{2}H/U$ for $Q=73$. (Note, after adjusting k, the absolute values of H and U become unimportant; the ratio $\beta=\sqrt{2}H/U$ is the relevant quantity). In Fig. 4, we fix $\beta=50$ and plot G_m vs Q. It is striking that both curves in Figs. 3 and 4 are peaked at certain β or Q. We will call these peaks also stochastic resonance. However, here the peaked function G is neither the amplitude of the output signal nor the SNR of the output. The quantity G shows how much the device in Fig. 1 can improve the signal quality better than linear systems. Especially, the peak in Fig. 4 is with respect to the quantity Q; that manifests a different type of "SR." Changing Q can change both signal amplitude and noise intensity. However, for a linear filter, it does not change SNR (Fig. 2), while for a nonlinear filter it does considerably change SNR and, consequently, change the quantity G. The existence of the optimal Q causes the SR-like response of the system to the input for the given β.

In Fig. 3 the peak position and height are indicated by $\hat{\beta}$ and $\hat{G}_m$, respectively, $\hat{G}_m$ and $\hat{\beta}$ are functions of Q. In Figs. 5(a) and 5(b) we plot $\hat{G}_m$ and $\hat{\beta}$ against Q, respectively. From the figures the following points are worth remarking. First, as $Q\to 0$ we find $\hat{G}_m \lesssim 1$; that is consistent with all the previous results obtained for model (1). Second, for small Q the $\hat{G}_m$-Q curve goes up very rapidly. This point is very useful for practical purposes. Helped by a linear filter with low quality factor, the SR device can enhance SNR rather effectively. Third, we can get $\hat{G}_m > 1$ at finite Q. Thus, by applying the set Fig. 1, we can exceed the limit of linear treatment and get output with SNR higher than that of the input. We find that G can be larger than 3, indicating that the SNR of the output can be higher than that of the input by a factor up to 10 if we compute SNR in terms of power spectra. Finally, $\hat{\beta}$ is a monotonously increasing function of Q. This fact is of crucial importance for application. Practically, it is most interesting to extract a weak signal from a strong noise. From Fig. 5(b) we know how to use a combination of linear and nonlinear devices to attack this goal. Actually, in this paper we are dealing with an input with SNR some ten times lower than that in Ref. [11] where the system (1) is investigated experimentally.

It is not difficult to intuitively understand the mechanism underlying Fig. 5 from the point of view of energy transfer. It is well known that under SR conditions the output signal may take some energy from the output noise. However, this energy transfer comes from the entire noise spectra [4,7,8]. In the case of white noise input this energy transfer is shared by widely distributed spectra and the noise reduction of the output at the signal frequency (i.e., the SFN) is limited. If we filter out some side-frequency-noise in the input, the SR device may more effectively transfer energy from the SFN to the signal and largely enhance SNR. For larger input noise (large β) we need larger Q to filter more side-spectra-noise for considerably reducing the SFN; that explains the monotonous behavior of Fig. 5(b).

In the previous discussions we fixed $f=f_0$ in Eqs. (2). However, it by no means indicates that we need a foreknowledge of the input frequency. Actually, we can find the input frequency in our experiment in a consistent manner. In Fig. 6 we fix $f_0=120$ Hz, $\beta=60$, $Q=73$, and vary f in Eq. (2b) in a wide range, assuming f_0 to be unknown. We plot R_{SR} and R_L against f where R_{SR} and R_L are measured at the given frequency f and maximal with respect to k. The two curves have peaks at $f=f_0$. Hence f_0 is "found" by the system itself. On the other hand, the peak of R_{SR} is much higher than that of R_L.

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